EXAMPLES OF VOA

- $\mathrm{Vir}_c = W(\mathfrak{sl}_2)$ $\qquad [W(\mathfrak{gl}_r)]$

$\mathrm{Vir}_c = \mathrm{span}(L_n \quad n \in \mathbb{Z}) \qquad c \in \mathbb{C}$

$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}$.

$L_n|0\rangle = 0 \quad n > 0$.

$\overset{\mathrm{Vir}_c}{\circlearrowleft}$

$V = \mathrm{span}(L_{-n_1} \cdots L_{-n_k}|0\rangle) \qquad k \geq 0, \quad n_1 > \cdots > n_k > 0$

$Y : V \longrightarrow \mathrm{End}\, V[[z^{\pm 1}]]$.

$T = L_{-1}|0\rangle \mapsto \sum \frac{L_n}{z^{n+2}} = T(z)$.

$L_{-n_1} \cdots L_{-n_k}|0\rangle \mapsto :\prod_{j=1}^{k} \frac{\partial_z^{n_i - 1}}{(n_i - 1)!} T(z): \leftarrow$ normal ordered product put positive modes to the right. $\qquad \star$

is a VOA structure.

- $\mathfrak{h} = \mathbb{C}^r$ Cartan for $\mathfrak{gl}_r$, $\langle \cdot, \cdot \rangle$ Killing form.

$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[[z^{\pm 1}]] \oplus \mathbb{C}K \qquad \underset{n \in \mathbb{Z}}{\xi \in \mathfrak{h}} \qquad \xi_n = \xi \otimes x^n$

$\hat{\mathfrak{h}}_- = \mathfrak{h} \otimes x^{-1}\mathbb{C}[x^{-1}]$.

$V = \mathrm{Sym}(\hat{\mathfrak{h}}_-).|0\rangle = \mathcal{H}(\mathfrak{gl}_r)$.

$\underline{Y} : V \longrightarrow \mathrm{End}\, V[[x^{\pm 1}]]$.

$\xi_{-1}|0\rangle \longmapsto \sum_{n \in \mathbb{Z}} \frac{\xi_n}{x^{n+1}}$

$\xi_{-n_1} \cdots \xi_{-n_k}|0\rangle \longmapsto$ reconstruction formula $\star$

- $\underbrace{W(\mathfrak{gl}_r)}_{c = r}$ $\qquad W^i = e_i(x^1_{-1}, \ldots x^r_{-1})|0\rangle$ (Miwa transform)

elementary symmetric.

where $x^1, \ldots, x^r$ is o.n basis on $\mathfrak{h}$

$W^i(z) = Y(W^i, z) = \sum_{n \in \mathbb{Z}} \frac{\textcircled{W^i_k}}{z^{k+i}}$ modes.

VOA strongly generated by W^i (freely)

$\mathrm{span}(W^{i_1}_{-k_1} \cdots W^{i_n}_{-k_n}|0\rangle \ \underset{k_1, \ldots, k_n > 0}{1 \leq i_1, \ldots, i_n \leq r}) = W(\mathfrak{gl}_r). \subseteq \mathcal{H}(\mathfrak{gl}_r)$.

$\mathcal{A}$ = (completed) associative algebra of modes

GRADED IDEALS in $\mathcal{A}$

For ② we want $\underline{\boxed{I} \subset [r] \times \mathbb{Z}}$ generating a graded Lie ideal

if $\mathcal{A}_I = \mathrm{span}(W^i_k \quad (i,k) \in I)$.

$[\mathcal{A}_I, \mathcal{A}_I] \subseteq \hbar \mathcal{A}\cdot\mathcal{A}_I$

(ideal in $\mathcal{A}$ generated $\mathcal{A}_I$ · $[\mathcal{A}\cdot\mathcal{A}_I, \mathcal{A}\cdot\mathcal{A}_I] \subseteq \hbar\, \mathcal{A}\cdot\mathcal{A}_I$

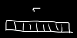
$\lambda_1 \geq \dots \geq \lambda_p \qquad \sum_{i=1}^{p} \lambda_i = r$

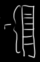
$\lambda[i] = \min\{m > 0 \mid \sum_{j=1}^{m} \lambda_j \geq i\}$

Lemma (B., Bouchard, Creutzig, Chidambaram, Noshchenko, 18)

For any such λ $\quad I_\lambda = \{(i,k) \mid k + \lambda[i] > 0\}$ generates a graded Lie ideal.

Ex (known before)

$\lambda =$  $\quad I_\lambda = \{k \geq 0,\ i=1,\dots,r\}$. vacuum ideal

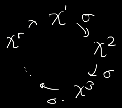
$\lambda =$  $\quad I_\lambda = \{k \geq -(i-1)\}$ conformal ideal

3 - FREE FIELD REPRESENTATION

→ $\sigma \in \mathrm{Aut}(\mathcal{W}(gl_r)) \cong \mathrm{Weyl}(gl_r) = \mathfrak{S}_r$

First treat $\sigma = (1 \cdots r)$ full cycle.

- Action of σ on $\mathfrak{h}$



- $\hat{\mathfrak{h}}_\sigma = \mathfrak{h} \otimes \mathbb{C}[[x^{\pm 1/r}]] \oplus \mathbb{C}K.$ $\qquad x = z^r$

action of σ $\qquad \sigma(\xi \otimes x^{n/r}) = \sigma(\xi)\, \theta^n \otimes x^{n/r}$ $\qquad \theta = \exp\left(\frac{2\pi i}{r}\right)$

- σ-Twisted representation of $\mathcal{W}(gl_r) = V$

$V \xrightarrow{Y_\sigma} \mathrm{End}\, V_\sigma[[x^{\pm 1/r}]]$ $\qquad V_\sigma = \mathrm{Sym}(\hat{\mathfrak{h}}_\sigma^-)|0\rangle \otimes \mathbb{C}[[\hbar]]$

$= \mathbb{C}[[x_1, x_2, x_3, \dots, \hbar]]$

(Fock space).

- Go to a diagonal basis for action of σ on $\mathfrak{h}$

$v^a = \sum_{j=1}^{r} \chi^j \theta^{-aj}$ $\qquad \sigma(v^a) = \theta^a v^a.$

$a = 1, \dots, r$ $\qquad \langle v^a, v^b \rangle = r\, \delta_{r | a+b}$

$\chi^j = \frac{1}{r} \sum_{a=1}^{r} \theta^{aj} v^a$

$Y_\sigma(v^a, z) = v^a(z) = \sum_{k \in \frac{a}{r} + \mathbb{Z}} \frac{J_{rk}}{x^{k+1}}$

$$J_k = \begin{cases} \hbar \partial_{x_k} & k > 0 \\ 0 & k = 0 \\ -k x_{-k} & k < 0 \end{cases}$$

$+$ reconstruction formula $\quad Y_\sigma (v^{a_1}_{-k_1} \cdots v^{a_m}_{-k_m} |0\rangle, z) = \cdots \quad *$

defines a σ-twisted representation of $\mathcal{W}(\mathfrak{gl}_r)$ on V_σ

restrict it to $W(\mathfrak{gl}_r) \subset \mathcal{H}(\mathfrak{gl}_r)$.

$$W^i(x) = \sum_{k \in \mathbb{Z}}{}' \frac{\boxed{W^i_k}}{x^{k+i}} \qquad \text{(untwisted)}$$

$\boxed{W^i_k} \longrightarrow$ differential operators.

COMPUTATION OF $W^i(x)$

$$\begin{aligned} W^i(x) &= : e_i (\chi^1(x), \ldots, \chi^r(x)) : \\ &= \sum_{1 \le p_1 < \cdots < p_i \le r}{}' : \chi^{p_1}(x) \cdots \chi^{p_i}(x) : \\ &= \frac{1}{r^i} \sum_{1 \le p_1 < \cdots < p_i \le r}{}' \theta^{-p_1 a_1 - \cdots - p_i a_i} : v^{a_1}(x) \cdots v^{a_i}(x) : \end{aligned}$$

$$\underbrace{v^{a_1}(z_1)\, v^{a_2}(z_2)} = : v^{a_1}(z_1)\, v^{a_2}(z_2) : + \underline{\hbar}\, \underbrace{\delta_{r | a_1 + a_2}\, \partial_{z_2} \left(\frac{(z_2/z_1)^{a_1/r}}{z_1 - z_2} \right)}.$$

more generally, Wick theorem:

$$W^i(x) = \frac{1}{r^i} \sum_{j=0}^{[i/2]}{}' \frac{i!}{2^j j! (i-2j)!} \sum_{1 \le a_{2j+1}, \ldots, a_i \le r}{}' \frac{\Psi^{(j)}_i (a_{2j+1}, \ldots, a_r)}{(\hbar x^{-2})^j} : \prod_{2j+1}^{i} v^{a_l}(x) :$$

$$\Psi^{(j)}_i = \frac{1}{i!} \sum_{m_1, \ldots, m_i = 0}^{\sigma - 1}{}' \prod_{l=1}^{j} \frac{\theta^{m_{2l-1} + m_{2l}}}{(\theta^{m_{2l}} - \theta^{m_{2l-1}})^2} \prod_{l=2j+1}^{i} \theta^{-m_l a_l} \in \mathbb{Q}$$

$j = 0, 1$ can be computed

diff op of degree i $\qquad \deg \hbar = 2.$
$\qquad \deg x_i = 1$

DILATON SHIFT

Choose $s > 0$

$$(x_k)_{k>0} \qquad \underset{\substack{\shortparallel \\ s x_{-s}}}{\underbrace{J_{-s}}} \longrightarrow J_{-s} - 1 \qquad \text{breaks homogeneity.}$$

$$W^i_{k | J_{-s} \to J_{-s} - 1} = \text{degree 0 terms} + \underbrace{x \qquad \hbar \partial_x} + O(2).$$

Tension : - no x (excludes, dpd on s, some i,k).
no deg 0 term.

- modes forming graded ideal : keep modes enough.

$$\boxed{W^i_k} = \hbar\,\partial_{\underline{x_{\pi(i,k)}}} + O(2)$$

$$\pi : I \xrightarrow{\sim} \mathbb{Z}_+$$

Thm (BBCCN 18). $\quad I_{r,s} = \{(i,k) \mid \underbrace{s(i-1)+rk}_{\pi(i,k)} > 0\}$.

Assume $s \in \{1,\dots,r+1\}$

$\underline{H_{\pi(i,k)}} = W^i_k\big|_{J_{-s}\to J_{-s}-1} = c_{ik}\,\hbar\,\partial_{x_{\pi(i,k)}} + O(2)$.

is an Airy structure iff $\boxed{r = \pm 1 \bmod s}$ ← $\boxed{\text{Basic Airy structures}}$

NB. if $\gcd(r,s) \neq 1$, π not a bijection. (cannot be Airy).

. simplest case excluded where (r,s) coprime is $(7,5)$

$I_{r,s}$ is not of the form I_λ
and in fact cannot be an ideal

$F_{0,3}[k_1,\dots,k_3]$ symmetric $\Leftarrow$ $I_{r,s}$ ideal

compute F_{03} explicitly : not symmetric unless $r = \pm 1 \bmod s$

SYMMETRIES

- $H_p \longrightarrow \mathcal{U} H_p \mathcal{U}^{-1}$ $\qquad \mathcal{U} \in \exp\left(\frac{1}{\hbar} W^{\hbar}_{\nu_\sigma}\right)$.

  $Z \longrightarrow \mathcal{U} Z$

  in general $\mathcal{U} H_p \mathcal{U}^{-1}$ is not an Airy structure

  (② still ok because conjugation

  ① $H_p = \underline{\hbar\partial_{x_p}} + O(2)$)

  $\partial_{x_p} \longrightarrow -x_p$ $\qquad$ does not respect ①.

  $x_p \longrightarrow \hbar\partial_{x_p}$

- $$\mathcal{U} = \exp\left[\sum_{\substack{p_1>0\\ p_2>0}}' \frac{1}{2}\,\frac{F_{0,2}[-p_1,-p_2]}{\hbar\,p_1 p_2}\, \overset{\hbar\partial_{x_{p_1}},\,\hbar\partial_{x_{p_2}}}{J_{p_1} J_{p_2}} + \sum_{p>0} \frac{F_{0,1}[-p]}{\hbar p}\, J_p\right]$$

$$= \exp\Big[\underbrace{\sum_{\substack{p_1>0\\ p_2>0}}' \frac{\hbar\, \partial_{x_{p_1}} \partial_{x_{p_2}}}{2} \frac{F_{0,2}[-p_1,-p_2]}{p_1 p_2}} + \underbrace{\sum_{p>0} \frac{F_{0,1}[-p]}{p}\, \partial_{x_p}} \Big]$$

BCH formula

$$X_p \longrightarrow X_p + \sum_q' \frac{F_{0,2}[-p,q]}{p\,q}\, \hbar\partial_{x_q}$$

$$\hbar\partial_{x_p} \to \hbar\partial_{x_p}.$$

action is always well-defined

$$X_p \longrightarrow \boxed{X_p} + \boxed{\frac{F_{0,1}[-p]}{p}}$$

not always well-defined

but $F_{0,1}[-p]=0 \quad p \leq s-1$

σ

$\mathcal{U} H_p \mathcal{U}^{-1}$ is well def and still forms an Airy structure

$$\frac{x = z^r}{F_{0,1},\ F_{0,2}}$$

ydx

$\omega_{0,2}$

$s = \min\{p \mid F_{0,1}[-p] \neq 0\}$

$\Rightarrow$ family of Airy str parametrized by $F_{0,1}, F_{0,2}$.

global curve $\longrightarrow$ local sp curve $\longleftrightarrow$ Airy str.

spec $\mathbb{C}[x,z] / x = z^r$

- $\sigma = (1 \cdots r)$ Galois sp

- $\sigma = \gamma_1 \cdots \gamma_p \longrightarrow \operatorname{spec} \mathbb{C}[x,z] / \prod_{\mu=1}^{p} (x - z^{r_\mu})$

$\gamma_i = r_i$-cycle, disjoint

$P(y,x) = 0 \qquad y \sim z^{\boxed{s_\mu} - r_\mu}$ on branch μ